Maths Assignment - 10

1(a) Let $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in W$.

$\alpha = 3$

$\therefore \alpha w = 3\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = (1, 1, 1) \notin W$.

$\therefore$ Not a subspace.

1(b) $W = \{(a, b, c) \mid b = 0 \text{ \& } a, b, c \in \mathbb{R}\}$

$(0, 0, 0) \in W$

$\therefore W$ is a non-empty subset of V.

Let $(a, 0, c)$ and $(x, 0, y) \in W$, $a, c, x, y \in \mathbb{R}$.

$\therefore (a, 0, c) + (x, 0, y) = (a+x, 0, c+y)$

$\in W$, $a+x, c+y \in \mathbb{R}$.

Let α be real.

$\alpha(a, 0, c) = (\alpha a, 0, \alpha c)$ $\quad \alpha a, \alpha c, \alpha x, \alpha y \in \mathbb{R}$.

$\alpha(x, 0, y) = (\alpha x, 0, \alpha y)$

$\therefore W$ is a subspace over $\mathbb{R}$.

1(c) $(0, 0, 0) \in W$.

$\therefore W$ is a ~~subspace~~ non-empty subset of V.

Let (x, x, x) and $(y, y, y) \in W$. $x, y \in \mathbb{R}$.

$\therefore (x, x, x) + (y, y, y) = (x+y, x+y, x+y)$

$\in W$, $x+y \in \mathbb{R}$.

Let α be real.

$\alpha(x, x, x) = (\alpha x, \alpha x, \alpha x)$ $\quad \alpha x, \alpha y \in \mathbb{R}$.

$\alpha(y, y, y) = (\alpha y, \alpha y, \alpha y)$

$\therefore W$ is a subspace over $\mathbb{R}$.

1(d) Let $w = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in W$.

Let $\alpha = 2$

$\therefore \alpha w = 2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (1,1,1) \notin W$.

$\therefore$ Not a subspace.

2) $W = \{ M_{m \times n} \mid m = n, \ tr(M) = 0 \}$.

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$.

$\therefore$ W is a non-empty subset of V.

Let A & $B \in W$

$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$.

$\therefore$ Let $\alpha, \beta \in \mathbb{R}$.

$$\alpha A + \beta B = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & -\alpha a \end{bmatrix} + \begin{bmatrix} \beta x & \beta y \\ \beta z & -\beta x \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a + \beta x & \alpha b + \beta y \\ \alpha c + \beta z & -\alpha a - \beta x \end{bmatrix} \in W.$$

$\therefore \alpha A + \beta B \in W \quad \forall \ A, B \in W, \ \alpha, \beta \in \mathbb{R}$.

$\therefore$ W is a subspace of V.

3) (a) $W = \{ f(x) : f(1) = 0 \}$

Taking $x - 1 \in W$.

$f(1) = 1 - 1 = 0$.

$\therefore$ W is a non-empty subset.

Let $\alpha, \beta \in \mathbb{R}$.

~~Let x_1 & $x_2 \in W$.~~

~~$\therefore f(x_1)$.~~

Let $ax_1 - a$, $bx_2 - b \in W$.

Now, $\alpha(ax_1 - a) + \beta(bx_2 - b)$

$= \alpha a x_1 - a\alpha + \beta b x_2 - b\beta$

$= 0$ $\{\because x_1 = 1 \ \& \ x_2 = 1\}$.

$\therefore$ W is a subspace over V.

3(b) $W = \{f(x) : f(3) = f(1)\}$

Taking $0(x) \in W$,

$\therefore$ W is non empty subset.

Now, Let $\alpha, \beta \in \mathbb{R}$.

Let $f(x), g(x) \in W$.

$\therefore (\alpha f + \beta g)(3) = (\alpha f + \beta g)(1)$

$\alpha f(3) + \beta g(3) = \alpha f(1) + \beta g(1)$

$\alpha f(1) + \beta g(1) = \alpha f(1) + \beta g(1)$ $\left\{\because \begin{array}{l} f(3) = f(1) \\ g(3) = g(1) \end{array}\right\}$.

$\therefore$ W is a subspace over V.

3(c) $W = \{f(x) : f(x) = -f(-x)\}$.

Taking $x^3 \in W$.

$\therefore$ W is a non-empty subset.

Now, Let $\alpha, \beta \in \mathbb{R}$.

Let $f(x), g(x) \in W$,

$\therefore (\alpha f + \beta g)(x) = -(\alpha f + \beta g)(-x)$

$\alpha f(x) + \beta g(x) = -(\alpha f + \beta g)(-x)$ $\left\{\because \begin{array}{l} f(x) = -f(-x) \\ g(x) = -g(-x) \end{array}\right\}$

$-(\alpha f + \beta g)(-x) = -(\alpha f + \beta g)(-x)$

W is a subspace over V.

4(a) $V = \{ P(x) \mid \deg(P) = 5 \}$.

$\therefore$ Let $A = x^5 + 1 \in V$

& $B = -x^5 \in V$

$A + B = 1 \notin V$.

$\therefore$ V is not closed under vector addition

so, Not a vector space.

4(b) (i) $V = \{ M_{n\times n} \mid M \text{ is a diagonal matrix} \}$

Vector Addition - $M_{n\times n} = \{A$ is a diagonal matrix$\}$.

$N_{n\times n} = \{B$ is a diagonal matrix$\}$.

$A + B = (M+N)_{n\times n}$

Scalar Multiplication = $\alpha A = \alpha M_{n\times n}$

Properties under vector Addition -

1) $B, A \in V$

$\therefore A + B \in V$.

closure Property.

2) Let $A, B, C \in V$.

Now, $(A+B)+C = \text{dia}(A+B) + \text{dia}(c)$

$= \text{diagonal}(A+B+c)$

$= \text{diagonal}(A) + \text{diagonal}(B+c)$

$= A + (B+c)$

Associative Property.

3) Let $A \in V$.

Now, $A + O = A = O + A$

O is a Null Matrix. & A is Diagonal Matrix.

$\therefore$ Additive identity is O.

4) Let $A \in V$.
then $-A \in V$.
$\therefore A + (-A) = 0$.
Additive inverse is $(-A)$.

5) Let $A, B \in V$.
then $A+B = B+A$
commutative Property.

Scalar Multiplication.

1) Let $\alpha \in \mathbb{R}$, $A \in V$.
$\therefore$ αA also $\in V$.

2) Let $\alpha, \beta \in \mathbb{R}$, $A \in V$.

$$(\alpha+\beta) A = (\alpha+\beta)\begin{bmatrix} a_{11} & 0 & 0 \cdots & 0 \\ 0 & a_{22} & 0 & 0 \\ \vdots & 0 & a_{33} & \\ 0 & 0 & & a_{nn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11}+\beta a_{11} & 0 & \cdots & 0 \\ \vdots & \alpha a_{22}+\beta a_{22} & \cdots & 0 \\ 0 & \vdots & \cdots & \alpha a_{nn}+\beta a_{nn} \end{bmatrix}$$

$$= \alpha A + \beta B.$$

3) Let $\alpha \in \mathbb{R}$, $A, B \in V$.

$$\alpha(A+B) = \alpha\left(\begin{bmatrix} a_{11} & 0 & 0 \cdots & 0 \\ \vdots & a_{22} & 0 \cdots & 0 \\ 0 & 0 & \ddots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ \vdots & b_{22} & \cdots & 0 \\ 0 & 0 & \ddots & b_{nn} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha a_{11} & 0 & \cdots & 0 \\ \vdots & \alpha a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & \alpha a_{nn} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & 0 & \cdots & 0 \\ 0 & \alpha b_{22} & \cdots & 0 \\ 0 & 0 & \ddots & \alpha b_{nn} \end{bmatrix}$$

$$= \alpha A + \alpha B$$

4) Let $\alpha, \beta \in \mathbb{R}$, $A \in V$.

$$(\alpha\beta) A = \alpha\beta\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & a_{nn} \end{bmatrix} = \alpha\begin{bmatrix} \beta a_{11} & 0 & \cdots & 0 \\ 0 & \beta a_{22} & \cdots & 0 \\ 0 & 0 & \ddots & \beta a_{nn} \end{bmatrix}$$

$$= \alpha(\beta A)$$

5) Let $A \in V$.
Then $1 \cdot A = A$.
Here, 1 is Unitary Matrix,

Now, Dimension of this $= n$

Also, $$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & & \vdots \\ \vdots & & & & \\ 0 & 0 & \cdots & & a_{nn} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} + \cdots$$
$$\cdots a_{nn}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \cdots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \\ 0 & & \ddots & 1 \end{bmatrix} \right\}$$

4(b) (ii) $V = \{ M_{n \times n} \mid M$ is a upper triangular matrix $\}$

Properties under Vector Addition-

1) Let $A, B \in V$.
then $A+B \in V$.
Closure Property

2) Let $A, B, C \in V$.
$$(A+B)+C = \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & a_{33} & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & & b_{nn} \end{bmatrix} \right) + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & c_{22} & \cdots & c_{2n} \\ & & \ddots & \\ 0 & \cdots & & c_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & \cdots \\ \vdots & \vdots & \\ \vdots & & \\ \vdots & \cdots & a_{nn}+b_{nn}+c_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ & \cdots & & a_{nn} \end{bmatrix} + \left(\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \ddots & \\ \vdots & & & \\ 0 & \cdots & 0 & b_{nn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & c_{nn} \end{bmatrix} \right)$$

$= A + (B + C)$

3) Let $A \in V$, $\exists\, O \in V$.

$\therefore A + O = A = O + A$

A is upper triangular & O is Null Matrix.

O is additive identity.

4) Let $A \in V$,

~~$\therefore -A \in V$.~~

Now $A + (-A) = O$

$-A$ is additive inverse of x.

5) Let $A, B \in V$.

$A + B = B + A$

commutative Property.

Scalar Multiplication-

1) Let $\alpha \in \mathbb{R}$, $A \in V$

$\therefore \alpha A \in V$.

2) Let $\alpha, \beta \in \mathbb{R}$, $A \in V$.

$(\alpha + \beta) A = \alpha A + \beta A$

3) Let $\alpha \in \mathbb{R}$, $A, B \in V$.

$\therefore \alpha (A + B) = \alpha A + \alpha B$

4) Let $\alpha, \beta \in \mathbb{R}$, $A \in V$.

Now, $(\alpha \beta) A = \alpha (\beta A)$

5) Let $A \in \mathbb{R}$.

$\therefore 1 . A = A$.

1 is unitary matrix.

A is upper triangular matrix.

~~Basi~~

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & & a_{nn}\end{bmatrix} = a_{11}\begin{bmatrix}1 & 0 & \cdots & 0\\ \vdots & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & & & 0\end{bmatrix} + a_{12}\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ \vdots & & \ddots & & \\ 0 & & 0 & & 0\end{bmatrix} + \cdots + a_{nn}\begin{bmatrix}0 & \cdots & 0\\ \vdots & 0 & 0\\ 0 & & 1\end{bmatrix}$$

$$\therefore \text{basis} = \left\{ \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & \cdots & & 0\\ \vdots & & \ddots & \\ 0 & & & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ \vdots & & 0 & \cdots & 0\\ 0 & & 0 & & 0\end{bmatrix}, \cdots, \begin{bmatrix}0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{bmatrix} \right\}$$

Dimension is $\dfrac{n(n+1)}{2}$

5) Let $x, y \in \mathbb{R}^2$.

$$\therefore (x, y) = c_1(2,5) + c_2(1,3)$$
$$= (2c_1 + c_2,\ 5c_1 + 3c_2)$$

$2c_1 + c_2 = x$

$5c_1 + 3c_2 = y$

$6c_1 + 3c_2 = 3x$

$c_1 = 3x - y$

$c_2 = x - 2(3x - y)$

$= x - 6x + 2y$

$c_2 = 2y - 5x$

$\therefore \{v_1, v_2\}$ is spanning set for $\mathbb{R}^2$.

$w = k_1 w_1 + k_2 (w_2)$

$(4, -7, 3) = k_1(1,2,0) + k_2(3,1,1)$

$= (k_1 + 3k_2,\ 2k_1 + k_2,\ k_2)$

$k_2 = 3$

$k_1 + 3k_2 = 4$	$2k_1 + k_2 = -7$
$k_1 = 4 - 9$	$2k_1 = -7 - 3$
$= -5$	$k_1 = -5$

k_1 & k_2 have unique values

$\therefore$ w belongs to span$\{w_1, w_2\}$.

6) $\text{span}\{v_1, v_2\} = \{c_1v_1 + c_2v_2 \;,\; c_1, c_2 \in \mathbb{R}\}$.

$= \{c_1(1,0,1) + c_2(0,1,1) \;,\; c_1, c_2 \in \mathbb{R}\}$

$= \{(c_1, c_2, c_1+c_2) \;,\; c_1, c_2 \in \mathbb{R}\}$

$\therefore$ subspace of $V = \mathbb{R}^3$ is $\{x, y, x+y \mid x, y \in \mathbb{R}\}$.

Now, $(1,1,-1) = c_1(1,0,1) + c_2(0,1,1)$

$= (c_1, c_2, c_1+c_2)$

$c_1 = 1 \qquad c_1 + c_2 = -1$

$c_2 = 1 \qquad c_1 = -2$

c_1 does not have unique values.

$\therefore$ $(1,1,-1)$ does not lie in this subspace.

7) $\text{span}\{v_1\} = \{c_1v_1 \;,\; c_1 \in \mathbb{R}\}$.

$= \{c_1(-1,1) \;;\; c_1 \in \mathbb{R}\}$.

$= \{(-c_1, c_1) \;,\; c_1 \in \mathbb{R}\}$.

This is span of $\{v_1\}$ & $\subseteq \mathbb{R}^2$.

8) $v = c_1v_1 + c_2v_2 + c_3v_3$

$(x, y, z) = c_1(-1,3,2) + c_2(1,-2,1) + c_3(2,1,1)$

$= (-c_1 + c_2 + 2c_3 \;,\; 3c_1 - 2c_2 + c_3 \;,\; 2c_1 + c_2 + c_3)$

$-c_1 + c_2 + 2c_3 = x$

$3c_1 - 2c_2 + c_3 = y$

$2c_1 + c_2 + c_3 = z$

$c_1 - 3c_2 = y - z$

$-3c_1 + c_3 = \cancel{x} - z$

$\cancel{-y + z - 3c_2 + c_2 + 2(x - z + 3c_1) = x}$

$\cancel{-2c_2 - y + z + 2x - 2z + 6(y - z + 3c_2) = x}$

$\cancel{-2c_2 - y - z + 2x + 6y - 6z + 18c_2 = x}$

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now, $\left[\begin{array}{ccc|c} -1 & 1 & 2 & x \\ 3 & -2 & 1 & y \\ 2 & 1 & 1 & z \end{array}\right]$

$R_2 \to R_2 + 3R_1$ & $R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & x \\ 0 & 1 & 7 & y+3x \\ 0 & 3 & 5 & z+2x \end{array}\right]$$

$R_3 \to R_3 - 3R_2$ $\left[\begin{array}{ccc|c} -1 & 1 & 2 & x \\ 0 & 1 & 7 & y+3x \\ 0 & 0 & -16 & -7x-3y+z \end{array}\right]$

Now, $c_3 = \dfrac{7x+3y-z}{16}$

$c_2 + 7c_3 = y + 3x$

$c_2 = y + 3x - \left(\dfrac{49x+21y-7z}{16}\right)$

$= \dfrac{-x-5y+7z}{16}$

$-c_1 + c_2 + 2c_3 = x$

$c_1 = c_2 + 2c_3 - x$

$= \dfrac{-x-5y+7z}{16} + \dfrac{14x+6y-2z}{16} - x$

$= \dfrac{-3x+y+5z}{16}$

$\therefore (x, y, z) = \dfrac{-3x+y+5z}{16}(-1, 3, 2) + \dfrac{-x-5y+7z}{16}(1, -2, 1)$

$+ \dfrac{7x+3y-z}{16}(2, 1, 1)$

9) U, V, W are L.I. vector.

then, $c_1 U + c_2 V + c_3 W = 0$

$\therefore c_1 = c_2 = c_3 = 0$

Now, $k_1(u+v) + k_2(u-v) + k_3(u-2v+w) = 0$.

$(k_1+k_2+k_3)u + (k_1-k_2-2k_3)v + k_3 w = 0$.

Now, $k_1+k_2+k_3 = 0$. , $k_3 = 0$.

$k_1-k_2-2k_3=0$.

$\therefore k_1+k_2=0$.

$k_1-k_2=0$

$2k_1=0 \Rightarrow k_1=0$.

$k_2=0$.

$\therefore$ $u+v$, $u-v$, $u-2v+w$ are also linearly independent.

10) $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, $R_5 \to R_5 - 2R_1$

$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_5$ $\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 3 & 5 & -3 & 1 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - \frac{R_2}{3}$, $R_4 \to R_4 - \frac{2}{3}R_2$

$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 3 & 5 & -3 & 1 \\ 0 & 0 & 4/3 & 0 & 5/3 \\ 0 & 0 & 8/3 & 0 & 10/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 \to R_4 - 2R_3$ $\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 3 & 5 & -3 & 1 \\ 0 & 0 & 4/3 & 0 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis $= \{(1,2,-1,3,4), (0,3,5,-3,1), (0,0,\frac{4}{3},0,\frac{5}{3})\}$.

11) $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$ $\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis = $\{(1,-2,5,-3), (0,7,-9,2)\}$.

Dimension = 2

Now, $B = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Extended Basis of W is $\{(1,-2,5,-3), (0,7,-9,2), (0,0,1,0), (0,0,0,1)\}$.

12) Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$

$\therefore AX = 0$

$\therefore \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + 3R_1$, $R_3 \to R_3 + 2R_1$

$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + \frac{R_2}{2}$ $\begin{bmatrix} -1 & 1 & 1 \\ \ldots \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \end{bmatrix}$

~~$-\frac{x_3}{2} \neq 0 \Rightarrow x_3 = 0.$~~

~~$-x_1 + x_2 + x_3 = 0$~~

~~$2x_2 + 3x_3 = 0$~~ ~~$x_1 = 0.$~~

~~$x_2 = 0$~~

~~$\therefore$ Soln of space of the system is origin only.~~

13) Now, for linearly dependent

$$|A| = 0.$$

$$A = \begin{bmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{bmatrix}$$

$$|A| = \lambda\left(\lambda^2 - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{-\lambda}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} + \frac{\lambda}{2}\right) = 0.$$

$$\lambda^3 - \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{1}{8} - \frac{1}{8} - \frac{\lambda}{4} = 0$$

$$\lambda^3 - \frac{3\lambda}{4} - \frac{1}{4} = 0.$$

$$4\lambda^3 - 3\lambda - 1 = 0 \Rightarrow (\lambda - 1)(2\lambda + 1)^2 = 0.$$

$$\lambda = 1, \frac{-1}{2} \text{ Ans}$$

14) $$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, Basis for row space = $\{(1, -3, 4, -2, 5, 4), (0, 0, 1, 3, -2, -6)\}$

Column space = $\{(1, -3, 4, -2, 5, 4), (2, -6, 9, -1, 9, 7)\}$

2) $W = \{ M_{n\times n} \mid tr(M) = 0 \}$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

$\therefore$ W is a non-empty subset of V.

Let $\alpha, \beta \in \mathbb{R}$

Let $A, B \in W$.

$\therefore tr(A) = 0 \;,\; tr(B) = 0$.

Now, $tr(\alpha A + \beta B)$

$= \alpha\, tr(A) + \beta\, tr(B)$

$= \alpha(0) + \beta(0)$

$= 0$.

$\therefore$ W is a subspace of V.

12) Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$

Now, $AX = 0$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1 \;,\; R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 + x_3 = 0$

$2x_2 + 3x_3 = 0$

$x_3 = \frac{-2x_2}{3} \Rightarrow x_2 = \frac{-3x_3}{2}$

$$x_1 = x_2 + x_3$$
$$= -\frac{3x_3}{2} + x_3$$
$$= \frac{-x_3}{2}$$

$\therefore \quad x_1 = \frac{-x_3}{2}$ and $x_2 = -\frac{3}{2}x_3$

This represents a line passing through the origin.